

Extended trial equation method for nonlinear partial differential equations

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Abstract

The main objective of this paper is to use the extended trial equation method to construct a series of some new analytic exact solutions for some nonlinear partial differential equations in mathematical physics. We will construct the exact solutions in many different functions such as hyperbolic function solutions, trigonometric function solutions, Jacobi elliptic functions solutions and rational functional solutions for the nonlinear partial differential equations when the balance number is real number via the Zhiber Shabat nonlinear differential equations. The balance number of this method is not constant as we shown in other methods but its changed by changing the trial equation derivative definition. This methods allowed us to construct many new type of exact solutions. We are shown by using the Maple software package that all obtained solutions are satisfied the original partial differential equations. The performance of this method is reliable, effective and powerful for solving more complicated nonlinear partial differential equations in mathematical physics.

Keywords: Nonlinear partial differential equations, Extend tial equation method, Exact solutions; Traveling wave transformation, Balance number, Soliton solutions, Jacobi elliptic functions.

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1. Introduction

The effort in finding exact solutions to nonlinear differential equations is important for the understanding of most nonlinear physical phenomena. For instance, the

nonlinear wave phenomena observed in fluid dynamics, plasma and optical fibers are often modeled by the bell shaped sech solutions and the kink shaped tanh solutions. In recent years, the exact solutions of non-linear PDEs have been investigated by many authors(see for example [1-28]) who are interested in non-linear physical phenomena. Many powerful methods have been presented by those authors such as the inverse scattering transform [1], the Backlund transform [2], Darboux transform [3], the generalized Riccati equation [4,5], the Jacobi elliptic function expansion method [6,7], Painlevé expansions method [8], the extended tanh-function method [9,10], the F-expansion method [11,12], the exp-function expansion method [13,14], the sub-ODE method [15,16], the extended sinh-cosh and sine-cosine methods [17,18], the (G'/G) -expansion method [19,20] and so on. Also there are many methods for finding the analytic approximate solutions for nonlinear partial differential equations such as the homotopy perturbation method [21,22], Adomain decomposition method [23,24], Variation iteration and homotopy analysis method [25,26]. There are many other methods for solving the nonlinear partial differential equations (see for example [27-35]). Recently Gurefe et al [36] have presented a direct method namely the extended trial equation method for solving the nonlinear partial differential equations. The main objective of this paper is to modify the extended trial equation method to construct a series of some new analytic exact solutions for some nonlinear partial differential equations in mathematical physics via the Zhiber Shabat nonlinear differential equations. In this present paper, we will construct the exact solutions in many different type of the roots of the trial equation. We will obtain many different kind of exact solutions in hyperbolic function solutions, trigonometric function solutions, Jacobi elliptic functions solutions and rational solutions. In this paper we get the balance number is not constant and changes by changing the trial equation derivative of the nonlinear partial differential equations.

2- Description of the extended trial equation method

Suppose that we have a nonlinear partial differential equation in the following form:

$$F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0, \quad (2.1)$$

where $u = u(x, t)$ is an unknown function, F is a polynomial in $u = u(x, t)$ and its partial derivatives, in which the highest order derivatives and nonlinear terms are

involved. Let us now give the main steps for solving Eq. (2.1) using the extended trial equation method as [36,37]:

Step 1. The traveling wave variable

$$u(x, t) = u(\xi), \quad \xi = x - \omega t, \tag{2.2}$$

where ω is a nonzero constant, the transformation (2.2) permits us reducing Eq. (2.1) to an ODE for $u = u(\xi)$ in the following form

$$P(u, -\omega u', u', \omega^2 u'', -\omega u'', u'', \dots) = 0, \tag{2.3}$$

where P is a polynomial of $u = u(\xi)$ and its total derivatives.

Step 2. Suppose the trial equation take the form:

$$u(\xi) = \sum_{i=0}^{\delta} \tau_i Y^i, \tag{2.4}$$

where Y satisfies the following nonlinear trial differential equation:

$$(Y')^2 = \Lambda(Y) = \frac{\Phi(Y)}{\Psi(Y)} = \frac{\xi_{\theta} Y^{\theta} + \xi_{\theta-1} Y^{\theta-1} + \dots + \xi_1 Y + \xi_0}{\zeta_{\varepsilon} Y^{\varepsilon} + \zeta_{\varepsilon-1} Y^{\varepsilon-1} + \dots + \zeta_1 Y + \zeta_0} \tag{2.5}$$

where ξ_i, ζ_j are constants to be determined later. Using (2.4) and (2.5), we have

$$u''(\xi) = \frac{\Phi'(Y)\Psi(Y) - \Phi(Y)\Psi'(Y)}{2\Psi^2(Y)} \left(\sum_{i=0}^{\delta} i \tau_i Y^{i-1} \right) + \frac{\Phi(Y)}{\Psi(Y)} \left(\sum_{i=0}^{\delta} i(i-1)\tau_i Y^{i-2} \right). \tag{2.6}$$

where $\Phi(Y), \Psi(Y)$ are polynomials in Y .

Step 3. Balancing the highest derivative term with the nonlinear term we can find the relations between δ, θ and ε . We can calculate some values of δ, θ and ε .

Step 4. Substituting the Eqs.(2.4)- (2.6) into (2.3) yields an polynomial $\Omega(y)$ of Y as following

$$\Omega(y) = \rho_s Y^s + \dots + \rho_1 Y + \rho_0 = 0 \tag{2.7}$$

Step 5. Setting the coefficients of the polynomial $\Omega(y)$ to be zero we, will yields a set of algebraic equations

$$\rho_i = 0, \quad i = 0, \dots, s. \tag{2.8}$$

Solving this system of algebraic equations to determine the values of $\xi_\theta, \xi_{\theta-1}, \dots, \xi_1, \xi_0, \zeta_\varepsilon, \zeta_{\varepsilon-1}, \dots, \zeta_1, \zeta_0$ and $\tau_\delta, \tau_{\delta-1}, \dots, \tau_1, \tau_0$.

Step 6. Reduce Eq.(2.5) to the elementary integral form:

$$\pm (\xi - \eta_0) = \int \frac{dY}{\sqrt{\Lambda(Y)}} = \int \sqrt{\frac{\Psi(Y)}{\Phi(Y)}} dY. \quad (2.9)$$

where η_0 is an arbitrary constant. Using a complete discrimination system for the polynomial to classify the roots of $\Phi(Y)$, we solve (2.9) with the help of software package such as Maple or Mathematica and classify the exact solutions to Eq.(2.3). In addition, we can write the exact traveling wave solutions to (2.1), respectively.

3. Extended trial equation method for nonlinear the Zhiber Shabat nonlinear differential equations

We start with the following nonlinear Zhiber Shabat differential equation:

$$u_{tx} + pe^u + qe^{-u} + re^{-2u} = 0 \quad (3.1)$$

where p, q and r are nonzero constants. The traveling wave variable (2.2) permits us converting equation (3.1) into the following ODE:

$$-\omega u'' + pe^u + qe^{-u} + re^{-2u} = 0. \quad (3.2)$$

If we use the transformation

$$v = e^u \quad (3.3)$$

The transformation (3.3) leads to write Eq.(3.2) in the following form:

$$-\omega(vv'' - v'^2) + pv^3 + qv + r = 0. \quad (3.4)$$

From Eqs.(2.4)-(2.9), we can write the following highest order nonlinear terms in order to determine the balance procedure:

$$v(\xi) = \tau_\delta Y^\delta + \tau_{\delta-1} Y^{\delta-1} + \dots, \quad (3.5)$$

$$v^3(\xi) = \tau_\delta^3 Y^{3\delta} + \dots, \quad (3.6)$$

and

$$(v')^2 = \frac{\xi_\theta \delta^2 \tau_\delta^2}{\zeta_\varepsilon} Y^{2\delta-2} (Y')^2 + \dots = \frac{\xi_\theta \delta^2 \tau_\delta^2}{\zeta_\varepsilon} Y^{2\delta+\theta-\varepsilon-2} + \dots \quad (3.7)$$

$$vv'' = W Y^{2\delta+\theta-\varepsilon-2} + \dots$$

From (3.5)-(3.7) lead to get the relation between δ, θ and ε as following

$$\theta = \varepsilon + \delta + 2 \quad (3.8)$$

Equation (3.8) has infinity solutions, consequently we suppose some of these solutions as following:

Case 1. In the special case if $\varepsilon = 0$ and $\delta = 1$ we get $\theta = 3$. Equations (2.4)-(2.9) lead to get

$$\begin{aligned} v &= \tau_0 + \tau_1 Y, \\ (v')^2 &= \frac{\tau_1^2 (\xi_3 Y^3 + \xi_2 Y^2 + \xi_1 Y + \xi_0)}{\zeta_0}, \\ v v'' &= \frac{\tau_1 (3\xi_3 Y^2 + 2\xi_2 Y + \xi_1)}{2\zeta_0}. \end{aligned} \tag{3.9}$$

Substituting Eqs.(3.9) into Eq.(3.4), we get a system of algebraic equations which can be solved to obtain the following results:

$$\begin{aligned} \xi_0 &= \frac{-1}{\omega \tau_1^2} (2q \tau_0 \zeta_0 - \omega \tau_0^2 \xi_2 + 4p \tau_0^3 \zeta_0 + r \zeta_0), & \xi_3 &= \frac{2p \zeta_0 \tau_1}{\omega}, \\ \xi_1 &= \frac{-2}{\omega \tau_1} (q \zeta_0 - \omega \tau_0 \xi_2 + 3p \tau_0^2 \zeta_0), \end{aligned} \tag{3.10}$$

where $\zeta_0, \xi_2, \omega, \tau_1$ and τ_0 are arbitrary constants. Substituting these results (3.10) into Eqs. (2.5) and (2.9), we have

$$\pm (\eta - \eta_0) = \int \frac{dY}{\sqrt{\frac{\xi_3}{\zeta_0} Y^3 + \frac{\xi_2}{\zeta_0} Y^2 + \frac{\xi_1}{\zeta_0} Y + \frac{\xi_0}{\zeta_0}}}, \tag{3.11}$$

Now will discuss the roots of the following equation

$$\begin{aligned} &\frac{2p\tau_1}{\omega} Y^3 + \frac{\xi_2}{\zeta_0} Y^2 - \frac{2}{\omega \zeta_0 \tau_1} (q \zeta_0 - \omega \tau_0 \xi_2 + 3p \tau_0^2 \zeta_0) Y \\ &- \frac{1}{\omega \zeta_0 \tau_1^2} (2q \tau_0 \zeta_0 - \omega \tau_0^2 \xi_2 + 4p \tau_0^3 \zeta_0 + r \zeta_0) = 0. \end{aligned} \tag{3.12}$$

To integrate equations (3.11) we must discuss the different cases of the roots of Eqs.(3.12) as following families:

Family 1. If equation (3.12) has three equal repeated roots, consequently we can write the (3.12) in the following form:

$$\begin{aligned} &\frac{2p\tau_1}{\omega} Y^3 + \frac{\xi_2}{\zeta_0} Y^2 - \frac{2}{\omega \zeta_0 \tau_1} (q \zeta_0 - \omega \tau_0 \xi_2 + 3p \tau_0^2 \zeta_0) Y \\ &- \frac{1}{\omega \zeta_0 \tau_1^2} (2q \tau_0 \zeta_0 - \omega \tau_0^2 \xi_2 + 4p \tau_0^3 \zeta_0 + r \zeta_0) - (Y - \alpha_1)^3 = 0. \end{aligned} \tag{3.13}$$

From equating the coefficients of Y of both sides of Eq.(3.13), we get a system of algebraic equations in $\zeta_0, \xi_2, \omega, \tau_1$ and τ_0 which can be solved by using the Maple software package to get the following results:

$$r = \mp \frac{2q}{9p} \sqrt{-3qp}, \quad \xi_2 = -3\alpha_1 \zeta_0, \quad \tau_0 = \frac{1}{2p} \left(-\alpha_1 \omega \pm \frac{2}{3} \sqrt{-3qp} \right), \quad \tau_1 = \frac{\omega}{2p}, \quad (3.14)$$

Equations (3.14), (3.10) and (3.11) lead to get:

$$\xi_0 = -\alpha_1^3 \zeta_0, \quad \xi_1 = 3\alpha_1^2 \zeta_0, \quad \xi_3 = \zeta_0, \quad (3.15)$$

where ζ_0 is an arbitrary constant and

$$\pm (\xi - \eta_0) = \int \frac{dY}{(Y - \alpha_1)^{3/2}} = \frac{-2}{\sqrt{Y - \alpha_1}}. \quad (3.16)$$

or

$$Y = \alpha_1 + \frac{4}{(x - \omega t - \eta_0)^2}. \quad (3.17)$$

Substituting the solutions (3.14), (3.15) and (3.17) into (3.9) we get the exact solution of Eq. (3.4) has the form:

$$v(x, t) = \pm \frac{1}{3p} \sqrt{-3qp} + \frac{4\omega}{2p(x - \omega t - \eta_0)^2} \quad (3.18)$$

Hence the exact solution of nonlinear Zhiber Shabat differential equation (3.1) take the following form:

$$u(x, t) = \ln \left(\left| \pm \frac{1}{3p} \sqrt{-3qp} + \frac{4\omega}{2p(x - \omega t - \eta_0)^2} \right| \right) \quad (3.19)$$

Family 2. If the equation (3.12) has two equal repeated roots α_1 and the third root is α_2 , $\alpha_1 \neq \alpha_2$, consequently we can write the (3.12) in the following form:

$$\begin{aligned} & \frac{2p\tau_1}{\omega} Y^3 + \frac{\xi_2}{\zeta_0} Y^2 - \frac{2}{\omega\zeta_0\tau_1} (q\zeta_0 - \omega\tau_0\xi_2 + 3p\tau_0^2\zeta_0) Y \\ & - \frac{1}{\omega\zeta_0\tau_1^2} (2q\tau_0\zeta_0 - \omega\tau_0^2\xi_2 + 4p\tau_0^3\zeta_0 + r\zeta_0) - (Y - \alpha_1)^2 (Y - \alpha_2) = 0. \end{aligned} \quad (3.20)$$

From equating the coefficients of Y of both sides of Eq.(3.20), we get a system of algebraic equations in $\zeta_0, \xi_2, \omega, \tau_1$ and τ_0 which can be solved by using the Maple software package to get the following results:

$$\begin{aligned} r &= -\frac{1}{18p} \{ D[12pq + 2\omega^2(\alpha_1 - \alpha_2)^2] + 2pq\omega\alpha_2 + 4pq\omega\alpha_1 + \omega^3\alpha_1(\alpha_1 - \alpha_2)^2 \}, \\ \xi_2 &= -2\alpha_1\zeta_0 - \alpha_2\zeta_0, \quad \tau_0 = \frac{D}{p}, \quad \tau_1 = \frac{\omega}{2p}. \end{aligned} \quad (3.21)$$

where $D = -\frac{\alpha_2\omega}{6} - \frac{\alpha_1\omega}{3} \pm \frac{1}{6}\sqrt{\omega^2(\alpha_1 - \alpha_2)^2 - 12pq}$. Equations (3.21), (3.10) and

(3.11) lead to get:

$$\xi_0 = -\alpha_1^2\alpha_2\zeta_0, \quad \xi_1 = \alpha_1(\alpha_1 + 2\alpha_2)\zeta_0, \quad \xi_3 = \zeta_0, \quad (3.22)$$

where ζ_0 is an arbitrary constant and if $\alpha_2 > \alpha_1$, we have

$$\pm(\xi - \eta_0) = \int \frac{dY}{(Y - \alpha_1)\sqrt{Y - \alpha_2}} = \frac{2}{\sqrt{\alpha_2 - \alpha_1}} \tan^{-1} \left[\frac{\sqrt{Y - \alpha_2}}{\sqrt{\alpha_2 - \alpha_1}} \right], \quad \alpha_2 > \alpha_1 \quad (3.23)$$

or

$$Y = \alpha_2 + (\alpha_2 - \alpha_1) \tan^2 \left[\frac{\sqrt{\alpha_2 - \alpha_1}}{2} (\xi - \eta_0) \right], \quad \alpha_2 > \alpha_1. \quad (3.24)$$

Substituting the solutions (3.24), (3.22) and (3.21) into (3.9), we get the exact solution of Eqs.(3.4) takes the form:

$$v(x, t) = \frac{D}{p} + \frac{\omega}{2p} \left\{ \alpha_2 + (\alpha_2 - \alpha_1) \tan^2 \left[\frac{\sqrt{\alpha_2 - \alpha_1}}{2} (x - 2kt - \eta_0) \right] \right\}, \quad (3.25)$$

Hence the exact solution of nonlinear Zhiber Shabat differential equation (3.1) take the following form:

$$u(x, t) = \ln \left(\left| \frac{D}{p} + \frac{\omega}{2p} \left\{ \alpha_2 + (\alpha_2 - \alpha_1) \tan^2 \left[\frac{\sqrt{\alpha_2 - \alpha_1}}{2} (x - 2kt - \eta_0) \right] \right\} \right| \right) \quad (3.26)$$

Also when $\alpha_1 > \alpha_2$, we have

$$Y = \alpha_1 + (\alpha_1 - \alpha_2) \operatorname{csch}^2 \left[\frac{\sqrt{\alpha_1 - \alpha_2}}{2} (\xi - \eta_0) \right], \quad \alpha_1 > \alpha_2. \quad (3.27)$$

Substituting the solutions (3.27), (3.22) and (3.21) into (3.9), we get the exact solution of Eqs.(3.4) takes the form:

$$v(x, t) = \frac{D}{p} + \frac{\omega}{2p} \left\{ \alpha_1 + (\alpha_1 - \alpha_2) \operatorname{csch}^2 \left[\frac{\sqrt{\alpha_1 - \alpha_2}}{2} (\xi - \eta_0) \right] \right\}, \quad (3.28)$$

Hence the exact solution of nonlinear Zhiber Shabat differential equation (3.1) take the following form:

$$u(x, t) = \ln \left(\left| \frac{D}{p} + \frac{\omega}{2p} \left\{ \alpha_1 + (\alpha_1 - \alpha_2) \operatorname{csch}^2 \left[\frac{\sqrt{\alpha_1 - \alpha_2}}{2} (\xi - \eta_0) \right] \right\} \right| \right) \quad (3.29)$$

Family 3. If the equation (3.12) has three different roots α_1 , α_2 and α_3 , consequently, we can write the (3.12) in the following form:

$$\frac{2p\tau_1}{\omega}Y^3 + \frac{\xi_2}{\zeta_0}Y^2 - \frac{2}{\omega\zeta_0\tau_1}(q\zeta_0 - \omega\tau_0\xi_2 + 3p\tau_0^2\zeta_0)Y - \frac{1}{\omega\zeta_0\tau_1^2}(2q\tau_0\zeta_0 - \omega\tau_0^2\xi_2 + 4p\tau_0^3\zeta_0 + r\zeta_0) - (Y - \alpha_1)(Y - \alpha_2)(Y - \alpha_3) = 0. \quad (3.30)$$

From equating the coefficients of Y of both sides of Eq.(3.30), we get a system of algebraic equations in $\zeta_0, \xi_2, \omega, \tau_1$ and τ_0 which can be solved by using the Maple software package to get the following results:

$$r = -\frac{1}{36p^2}\{D[24pq - 4\omega^2(\alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_2\alpha_1 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2)] + 4pq\omega(\alpha_1 + \alpha_2 + \alpha_3) + \omega^3(\alpha_1^2\alpha_2 + \alpha_2^2\alpha_1 + \alpha_2^2\alpha_3 + \alpha_3^2\alpha_2 + \alpha_1^2\alpha_3 + \alpha_3^2\alpha_1 - 6\alpha_1\alpha_2\alpha_3)\},$$

$$\xi_2 = -\zeta_0(\alpha_1 + \alpha_2 + \alpha_3), \quad \tau_0 = \frac{D}{p}, \quad \tau_1 = \frac{\omega}{2p}, \quad (3.31)$$

where

$$D = -\frac{\omega}{6}(\alpha_1 + \alpha_2 + \alpha_3) \pm \frac{1}{6}\sqrt{\omega^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_1\alpha_3 - \alpha_2\alpha_3 - \alpha_2\alpha_1) - 12pq}.$$

Equations (3.31), (3.10) and (3.11) lead to get:

$$\xi_0 = -\alpha_1\alpha_3\alpha_2\zeta_0, \quad \xi_1 = (\alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_1\alpha_2)\zeta_0, \quad \xi_3 = \zeta_0, \quad (3.32)$$

where ζ_0 is an arbitrary constant and if $\alpha_3 > \alpha_2 > \alpha_1$, we have

$$\pm(\xi - \eta_0) = \int \frac{dY}{\sqrt{(Y - \alpha_1)(Y - \alpha_2)(Y - \alpha_3)}} = -\frac{2}{\sqrt{\alpha_3 - \alpha_1}} \text{EllipticF} \left[\frac{\sqrt{Y - \alpha_1}}{\sqrt{\alpha_2 - \alpha_1}}, \sqrt{\frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3}} \right], \quad (3.33)$$

or

$$Y = \alpha_1 + (\alpha_2 - \alpha_1) \text{sn}^2 \left[\frac{\sqrt{\alpha_3 - \alpha_1}}{2}(\xi - \eta_0), \sqrt{\frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3}} \right], \quad (3.33)$$

Substituting the solutions (3.33), (3.31) and (3.32) into (3.9), we get the exact solutions of Eqs.(3.4) takes the form:

$$v(x,t) = \frac{D}{p} + \frac{\omega}{2p} \left\{ \alpha_1 + (\alpha_2 - \alpha_1) \text{sn}^2 \left[\frac{\sqrt{\alpha_3 - \alpha_1}}{2}(\xi - \eta_0), \sqrt{\frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3}} \right] \right\}, \quad (3.34)$$

Hence the exact solution of nonlinear Zhiber Shabat differential equation (3.1) take the following form:

$$u(x,t) = \ln \left(\left| \frac{D}{p} + \frac{\omega}{2p} \{ \alpha_1 + (\alpha_2 - \alpha_1) \operatorname{sn}^2 \left[\frac{\sqrt{\alpha_3 - \alpha_1}}{2} (\xi - \eta_0), \sqrt{\frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3}} \right] \} \right| \right) \quad (3.35)$$

Family 4. If the equation (3.12) has one real roots α_1 and two imaginary roots $\alpha_2 = N_1 + iN_2$ $\alpha_3 = N_1 - iN_2$, N_1, N_2 are real numbers, consequently we can write the (3.12) in the following form:

$$\frac{2p\tau_1}{\omega} Y^3 + \frac{\xi_2}{\zeta_0} Y^2 - \frac{2}{\omega\zeta_0\tau_1} (q\zeta_0 - \omega\tau_0\xi_2 + 3p\tau_0^2\zeta_0) Y - \frac{1}{\omega\zeta_0\tau_1^2} (2q\tau_0\zeta_0 - \omega\tau_0^2\xi_2 + 4p\tau_0^3\zeta_0 + r\zeta_0) - (Y - \alpha_1)(Y^2 - 2N_1Y + N_1^2 + N_2^2) = 0. \quad (3.36)$$

From equating the coefficients of Y of both sides of Eq.(3.36), we get a system of algebraic equations in $\zeta_0, \xi_2, \omega, \tau_1$ and τ_0 which can be solved by using the Maple software package to get the following results:

$$r = -\frac{1}{18p^2} \{ D[12pq - \omega^2(4N_1\alpha_1 + 6N_2^2 - 2N_1^2 - 2\alpha_1^2)] + 2pq\omega(2N_1 + \alpha_1) + \omega^3(N_1^3 + N_2^2N_2 + \alpha_1^2N_1 - 4N_2^2\alpha_1 - 2N_1^2\alpha_1) \},$$

$$\xi_2 = -\zeta_0(2N_1 + \alpha_1), \quad \tau_0 = \frac{D}{p}, \quad \tau_1 = \frac{\omega}{2p}, \quad (3.37)$$

where $D = -\frac{\omega}{6}(2N_1 + \alpha_1) \pm \frac{1}{6} \sqrt{\omega^2(N_1^2 + \alpha_1^2 - 3N_2^2 - 2\alpha_1N_1) - 12pq}$.

Equations (3.37), (3.10) and (3.11) lead to get:

$$\xi_0 = -\alpha_1\zeta_0(N_1^2 + N_2^2), \quad \xi_1 = (N_1^2 + N_2^2 + 2\alpha_1N_1)\zeta_0, \quad \xi_3 = \zeta_0, \quad (3.38)$$

where ζ_0 is an arbitrary constant. With the help of Maple software package the integration of Eq.(3.11) in this family takes the following form:

$$\pm (\xi - \eta_0) = \int \frac{dY}{\sqrt{(Y - \alpha_1)(Y^2 - 2N_1Y + N_1^2 + N_2^2)}} = \frac{2}{\sqrt{N_1 + iN_2 - \alpha_1}} \operatorname{EllipticF} \left[\frac{\sqrt{Y - \alpha_1}}{\sqrt{N_1 - iN_2 - \alpha_1}}, \sqrt{\frac{N_1 - iN_2 - \alpha_1}{N_1 + iN_2 - \alpha_1}} \right], \quad (3.39)$$

or

$$Y = \alpha_1 + (N_1 - iN_2 - \alpha_1) \operatorname{sn}^2 \left[\frac{\sqrt{N_1 + iN_2 - \alpha_1}}{2} (\xi - \eta_0), \sqrt{\frac{N_1 - iN_2 - \alpha_1}{N_1 + iN_2 - \alpha_1}} \right]. \quad (3.40)$$

Substituting the solutions (3.40), (3.38) and (3.37) into (3.9), we get the exact solutions of Eqs. (3.4) takes the form:

$$v(x,t) = \frac{D}{p} + \frac{\omega}{2p} \left\{ \alpha_1 + (N_1 - iN_2 - \alpha_1) sn^2 \left[\frac{\sqrt{N_1 + iN_2 - \alpha_1}}{2} (\xi - \eta_0), \sqrt{\frac{N_1 - iN_2 - \alpha_1}{N_1 + iN_2 - \alpha_1}} \right] \right\}, \quad (3.41)$$

Hence the exact solution of nonlinear Zhiber Shabat differential equation (3.1) take the following form:

$$u(x,t) = \ln \left(\frac{D}{p} + \frac{\omega}{2p} \left\{ \alpha_1 + (N_1 - iN_2 - \alpha_1) sn^2 \left[\frac{\sqrt{N_1 + iN_2 - \alpha_1}}{2} (\xi - \eta_0), \sqrt{\frac{N_1 - iN_2 - \alpha_1}{N_1 + iN_2 - \alpha_1}} \right] \right\} \right) \quad (3.42)$$

Case 2. In the special case if $\varepsilon = 0$ and $\theta = 4$, we get $\delta = 2$. Equations (2.4)-(2.9) lead to get

$$v(\xi) = \tau_0 + \tau_1 Y + \tau_2 Y^2, \quad (3.43)$$

$$(v')^2 = \frac{(\tau_1 + 2\tau_2 Y)^2 (\xi_4 Y^4 + \xi_3 Y^3 + \xi_2 Y^2 + \xi_1 Y + \xi_0)}{\zeta_0},$$

$$v'' = \frac{\tau_1 (4\xi_4 Y^3 + 3\xi_3 Y^2 + 2\xi_2 Y + \xi_1)}{2\zeta_0} + \frac{\tau_2 (6\xi_4 Y^4 + 5\xi_3 Y^3 + 4\xi_2 Y^2 + 3\xi_1 Y + 3\xi_0)}{\zeta_0}$$

Substituting Eq. (3.43) into Eqs. (3.4) and setting the coefficient Y to be zero, we get a system of algebraic equations which can be solved to obtain the following results:

$$\xi_0 = \frac{p^2 \zeta_0^2}{4\xi_4 \omega^2 (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[\omega^2 \tau_0^2 \xi_3^4 - 16\omega \tau_0^3 \xi_3^2 \xi_4 p \zeta_0 - 8\tau_0 \xi_4 \zeta_0 \omega \xi_3^2 q \right. \\ \left. + 64\tau_0^2 \xi_4^2 p \zeta_0^2 q + 64\tau_0^4 \xi_4^2 p^2 \zeta_0^2 + 64\tau_0 \xi_4^2 p \zeta_0^2 r - 4\omega \xi_3^2 r \zeta_0 \xi_4 \right]$$

$$\xi_1 = \frac{\xi_3 p \zeta_0}{2\xi_4 \omega (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[\omega^2 \tau_0 \xi_3^4 - 4\omega \xi_3^2 q \xi_4 \zeta_0 - 16\omega \xi_3^2 \xi_4 \tau_0^2 p \zeta_0 \right. \\ \left. + 32q \tau_0 \zeta_0^2 p \xi_4^2 + 64p^2 \tau_0^3 \zeta_0^2 \xi_4^2 + 16r \zeta_0^2 \xi_4^2 p \right]$$

$$\xi_2 = \frac{1}{4\xi_4 \omega^2 (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[-12\omega^2 \xi_3^4 \tau_0 p \xi_4 \zeta_0 + \omega^3 \xi_3^6 - 16\xi_4^2 p \zeta_0^2 \omega \xi_3^2 q \right. \\ \left. + 128\xi_4^3 p^2 \zeta_0^3 q \tau_0 + 256\xi_4^3 p^3 \zeta_0^3 \tau_0^3 + 64\xi_4^3 p^2 \zeta_0^3 r \right]$$

$$\tau_1 = \frac{\xi_3 \omega}{p \zeta_0}, \quad \tau_2 = \frac{2\xi_4 \omega}{p \zeta_0} \quad (3.44)$$

where $\zeta_0, \xi_3, \xi_4 p, q$ and r are arbitrary constants. Substituting these results (3.44) into Eqs. (2.5) and (2.9), we have

$$\pm (\xi - \eta_0) = \int \frac{dY}{\sqrt{\frac{\xi_4}{\zeta_0} Y^4 + \frac{\xi_3}{\zeta_0} Y^3 + \frac{\xi_2}{\zeta_0} Y^2 + \frac{\xi_1}{\zeta_0} Y + \frac{\xi_0}{\zeta_0}}}, \quad (3.45)$$

Now we will discuss the roots of the following equation

$$\begin{aligned} & \frac{\xi_4}{\zeta_0} Y^4 + \frac{\xi_3}{\zeta_0} Y^3 + \frac{1}{4\xi_4 \omega^2 \zeta_0 (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[-12\omega^2 \xi_3^4 \tau_0 p \xi_4 \zeta_0 + \omega^3 \xi_3^6 - 16\xi_4^2 p \zeta_0^2 \omega \xi_3^2 q \right. \\ & + 128\xi_4^3 p^2 \zeta_0^3 q \tau_0 + 256\xi_4^3 p^3 \zeta_0^3 \tau_0^3 + 64\xi_4^3 p^2 \zeta_0^3 r \left. \right] Y^2 + \frac{\xi_3 p}{2\xi_4 \omega (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[\omega^2 \tau_0 \xi_3^4 \right. \\ & - 4\omega \xi_3^2 q \xi_4 \zeta_0 - 16\omega \xi_3^2 \xi_4 \tau_0^2 p \zeta_0 + 32q \tau_0 \zeta_0^2 p \xi_4^2 + 64p^2 \tau_0^3 \zeta_0^2 \xi_4^2 + 16r \zeta_0^2 \xi_4^2 p \left. \right] Y \\ & + \frac{p^2 \zeta_0^2}{4\xi_4 \omega^2 (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[\omega^2 \tau_0^2 \xi_3^4 - 16\omega \tau_0^3 \xi_3^2 \xi_4 p \zeta_0 - 8\tau_0 \xi_4 \zeta_0 \omega \xi_3^2 q + 64\tau_0^2 \xi_4^2 p \zeta_0^2 q \right. \\ & \left. + 64\tau_0^4 \xi_4^2 p^2 \zeta_0^2 + 64\tau_0 \xi_4^2 p \zeta_0^2 r - 4\omega \xi_3^2 r \zeta_0 \xi_4 \right] = 0. \end{aligned} \quad (3.46)$$

To integrate equations (3.46), we must discuss the different cases of the roots of Eqs.(3.46) as the following families:

Family 5. If equation (3.46) has four equal repeated roots α_1 , consequently we can write the (3.46) in the following form:

$$\begin{aligned} & \frac{\xi_4}{\zeta_0} Y^4 + \frac{\xi_3}{\zeta_0} Y^3 + \frac{1}{4\xi_4 \omega^2 \zeta_0 (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[-12\omega^2 \xi_3^4 \tau_0 p \xi_4 \zeta_0 + \omega^3 \xi_3^6 - 16\xi_4^2 p \zeta_0^2 \omega \xi_3^2 q \right. \\ & + 128\xi_4^3 p^2 \zeta_0^3 q \tau_0 + 256\xi_4^3 p^3 \zeta_0^3 \tau_0^3 + 64\xi_4^3 p^2 \zeta_0^3 r \left. \right] Y^2 + \frac{\xi_3 p}{2\xi_4 \omega (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[\omega^2 \tau_0 \xi_3^4 \right. \\ & - 4\omega \xi_3^2 q \xi_4 \zeta_0 - 16\omega \xi_3^2 \xi_4 \tau_0^2 p \zeta_0 + 32q \tau_0 \zeta_0^2 p \xi_4^2 + 64p^2 \tau_0^3 \zeta_0^2 \xi_4^2 + 16r \zeta_0^2 \xi_4^2 p \left. \right] Y \\ & + \frac{p^2 \zeta_0^2}{4\xi_4 \omega^2 (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[\omega^2 \tau_0^2 \xi_3^4 - 16\omega \tau_0^3 \xi_3^2 \xi_4 p \zeta_0 - 8\tau_0 \xi_4 \zeta_0 \omega \xi_3^2 q + 64\tau_0^2 \xi_4^2 p \zeta_0^2 q \right. \\ & \left. + 64\tau_0^4 \xi_4^2 p^2 \zeta_0^2 + 64\tau_0 \xi_4^2 p \zeta_0^2 r - 4\omega \xi_3^2 r \zeta_0 \xi_4 \right] - (Y - \alpha_1)^4 = 0. \end{aligned} \quad (3.47)$$

From equating the coefficients of Y of both sides of Eq.(3.47), we get a system of algebraic equations in $\zeta_0, \xi_3, \xi_4, \tau_0$ and r, ω which can be solved by using the Maple software package to get the following results:

$$r = -\frac{2q}{3p} (-2\alpha_1^2 \omega + D), \quad \xi_3 = -4\alpha_1 \zeta_0, \quad \xi_4 = \zeta_0, \quad \tau_0 = \frac{D}{p}. \quad (3.48)$$

where $D = 2\alpha_1^2 \omega \pm \frac{\sqrt{-3pq}}{3}$.

Equations (3.48), (3.44) and (3.45) leads to get:

$$\xi_0 = \zeta_0 \alpha_1^4, \quad \xi_1 = -4\zeta_0 \alpha_1^3, \quad \xi_2 = 6\zeta_0 \alpha_1^2, \quad \tau_1 = -\frac{4\alpha_1 \omega}{p}, \quad \tau_2 = \frac{2\omega}{p} \quad (3.49)$$

where ζ_0 is an arbitrary constant and

$$\pm(\eta - \eta_0) = \int \frac{dY}{(Y - \alpha_1)^2} = \frac{-1}{Y - \alpha_1}, \quad (3.50)$$

or

$$Y = \alpha_1 \mp \frac{4}{(x - \omega t - \eta_0)}. \quad (3.51)$$

Substituting the solutions (3.51), (3.49) and (3.48) into (3.43), we get the exact solution of Eqs.(3.4) takes the form:

$$v(\xi) = \frac{D}{p} - \frac{4\alpha_1 \omega}{p} \left[\alpha_1 \mp \frac{4}{(x - \omega t - \eta_0)} \right] + \frac{2\omega}{p} \left[\alpha_1 \mp \frac{4}{(x - \omega t - \eta_0)} \right]^2 \quad (3.52)$$

Hence the exact solution of nonlinear Zhiber Shabat differential equation (3.1) take the following form:

$$u(x, t) = \ln \left(\left[\frac{D}{p} - \frac{4\alpha_1 \omega}{p} \left[\alpha_1 \mp \frac{4}{(x - \omega t - \eta_0)} \right] + \frac{2\omega}{p} \left[\alpha_1 \mp \frac{4}{(x - \omega t - \eta_0)} \right]^2 \right] \right). \quad (3.52)$$

Family 6. If the equation (3.46) has two equal repeated roots α_1, α_2 and $\alpha_1 \neq \alpha_2$ consequently we can write the (3.46) in the following form:

$$\begin{aligned} & \frac{\xi_4}{\zeta_0} Y^4 + \frac{\xi_3}{\zeta_0} Y^3 + \frac{1}{4\xi_4 \omega^2 \zeta_0 (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[-12\omega^2 \xi_3^4 \tau_0 p \xi_4 \zeta_0 + \omega^3 \xi_3^6 - 16\xi_4^2 p \zeta_0^2 \omega \xi_3^2 q \right. \\ & + 128\xi_4^3 p^2 \zeta_0^3 q \tau_0 + 256\xi_4^3 p^3 \zeta_0^3 \tau_0^3 + 64\xi_4^3 p^2 \zeta_0^3 r \left. \right] Y^2 + \frac{\xi_3 p}{2\xi_4 \omega (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[\omega^2 \tau_0 \xi_3^4 \right. \\ & - 4\omega \xi_3^2 q \xi_4 \zeta_0 - 16\omega \xi_3^2 \xi_4 \tau_0^2 p \zeta_0 + 32q \tau_0 \zeta_0^2 p \xi_4^2 + 64p^2 \tau_0^3 \zeta_0^2 \xi_4^2 + 16r \zeta_0^2 \xi_4^2 p \left. \right] Y \\ & + \frac{p^2 \zeta_0^2}{4\xi_4 \omega^2 (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[\omega^2 \tau_0^2 \xi_3^4 - 16\omega \tau_0^3 \xi_3^2 \xi_4 p \zeta_0 - 8\tau_0 \xi_4 \zeta_0 \omega \xi_3^2 q + 64\tau_0^2 \xi_4^2 p \zeta_0^2 q \right. \\ & \left. + 64\tau_0^4 \xi_4^2 p^2 \zeta_0^2 + 64\tau_0 \xi_4^2 p \zeta_0^2 r - 4\omega \xi_3^2 r \zeta_0 \xi_4 \right] - (Y - \alpha_1)^2 (Y - \alpha_2)^2 = 0. \end{aligned} \quad (3.53)$$

From equating the coefficients of Y of both sides of Eq.(3.53), we get a system of algebraic equations in $\zeta_0, \xi_3, \xi_4, \tau_0$ and r, ω which can be solved by using the Maple software package to get the following results:

$$r = -\frac{1}{p^2} \{ D[\omega^2(\alpha_1 - \alpha_2)^4 + 6pq] + 8\omega^2\alpha_1^2\alpha_2^4 + 8\omega^2\alpha_1^4\alpha_2^2 - pq\omega\alpha_2^2 - pq\omega\alpha_1^2 - 2\omega^3\alpha_1\alpha_2^5 - 2\omega^3\alpha_2\alpha_1^5 - 12\omega^3\alpha_1^3\alpha_2^3 - 10pq\omega\alpha_1\alpha_2 \}, \quad (3.54)$$

$$\xi_3 = -2\zeta_0(\alpha_1 + \alpha_2), \quad \xi_4 = \zeta_0, \quad \tau_0 = \frac{D}{p},$$

where $D = \frac{\omega}{6}(\alpha_1^2 + \alpha_2^2 + 10\alpha_1\alpha_2) \pm \frac{1}{6}\sqrt{\omega^2(\alpha_1 - \alpha_2)^4 - 12pq}$. Equations (3.54) ,

(3.44) and (3.45) leads to get:

$$\xi_0 = \zeta_0\alpha_1^2\alpha_2^2, \quad \xi_1 = -2\zeta_0(\alpha_1^2\alpha_2 + \alpha_2^2\alpha_1), \quad \xi_2 = \zeta_0(\alpha_1^2 + \alpha_2^2 + 4\alpha_1\alpha_2),$$

$$\tau_1 = -\frac{2\omega}{p}(\alpha_1 + \alpha_2), \quad \tau_2 = \frac{2\omega}{p} \quad (3.55)$$

where ζ_0 is an arbitrary constant and

$$\pm(\xi - \eta_0) = \int \frac{dY}{(Y - \alpha_1)(Y - \alpha_2)} = \frac{1}{\alpha_1 - \alpha_2} \ln \left| \frac{Y - \alpha_1}{Y - \alpha_2} \right| \quad (3.56)$$

Or

$$Y = \frac{-\alpha_1 + \alpha_2 e^{\pm(\alpha_1 - \alpha_2)(\xi - \eta_0)}}{-1 + e^{\pm(\alpha_1 - \alpha_2)(\xi - \eta_0)}} \quad (3.57)$$

Substituting the solutions (3.57), (3.55) and (3.54) into (3.43), we get the exact solution of Eqs.(3.4) takes the form:

$$v(\xi) = \frac{D}{p} - \frac{2\omega}{p}(\alpha_1 + \alpha_2) \left[\frac{-\alpha_1 + \alpha_2 e^{\pm(\alpha_1 - \alpha_2)(\xi - \eta_0)}}{-1 + e^{\pm(\alpha_1 - \alpha_2)(\xi - \eta_0)}} \right] + \frac{2\omega}{p} \left[\frac{-\alpha_1 + \alpha_2 e^{\pm(\alpha_1 - \alpha_2)(\xi - \eta_0)}}{-1 + e^{\pm(\alpha_1 - \alpha_2)(\xi - \eta_0)}} \right]^2 \quad (3.58)$$

Hence the exact solution of nonlinear Zhiber Shabat differential equation (3.1) take the following form:

$$u(x, t) = \ln \left(\left(\frac{D}{p} - \frac{2\omega}{p}(\alpha_1 + \alpha_2) \left[\frac{-\alpha_1 + \alpha_2 e^{\pm(\alpha_1 - \alpha_2)(\xi - \eta_0)}}{-1 + e^{\pm(\alpha_1 - \alpha_2)(\xi - \eta_0)}} \right] + \frac{2\omega}{p} \left[\frac{-\alpha_1 + \alpha_2 e^{\pm(\alpha_1 - \alpha_2)(\xi - \eta_0)}}{-1 + e^{\pm(\alpha_1 - \alpha_2)(\xi - \eta_0)}} \right]^2 \right) \right) \quad (3.59)$$

Family 7. If equation (3.46) has four different roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ consequently we can write the (3.46) in the following form:

$$\begin{aligned} & \frac{\xi_4}{\zeta_0} Y^4 + \frac{\xi_3}{\zeta_0} Y^3 + \frac{1}{4\xi_4 \omega^2 \zeta_0 (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[-12\omega^2 \xi_3^4 \tau_0 p \xi_4 \zeta_0 + \omega^3 \xi_3^6 - 16\xi_4^2 p \zeta_0^2 \omega \xi_3^2 q \right. \\ & + 128\xi_4^3 p^2 \zeta_0^3 q \tau_0 + 256\xi_4^3 p^3 \zeta_0^3 \tau_0^3 + 64\xi_4^3 p^2 \zeta_0^3 r \left. \right] Y^2 + \frac{\xi_3 p}{2\xi_4 \omega (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[\omega^2 \tau_0 \xi_3^4 \right. \\ & - 4\omega \xi_3^2 q \xi_4 \zeta_0 - 16\omega \xi_3^2 \xi_4 \tau_0^2 p \zeta_0 + 32q \tau_0 \zeta_0^2 p \xi_4^2 + 64p^2 \tau_0^3 \zeta_0^2 \xi_4^2 + 16r \zeta_0^2 \xi_4^2 p \left. \right] Y \\ & + \frac{p^2 \zeta_0^2}{4\xi_4 \omega^2 (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[\omega^2 \tau_0^2 \xi_3^4 - 16\omega \tau_0^3 \xi_3^2 \xi_4 p \zeta_0 - 8\tau_0 \xi_4 \zeta_0 \omega \xi_3^2 q + 64\tau_0^2 \xi_4^2 p \zeta_0^2 q \right. \\ & + 64\tau_0^4 \xi_4^2 p^2 \zeta_0^2 + 64\tau_0 \xi_4^2 p \zeta_0^2 r - 4\omega \xi_3^2 r \zeta_0 \xi_4 \left. \right] - (Y - \alpha_1)(Y - \alpha_2)(Y - \alpha_3)(Y - \alpha_4) = 0. \end{aligned} \tag{3.60}$$

From equating the coefficients of Y of both sides of Eq.(3.60) , we get a system of algebraic equations in ζ_0 , ξ_3 , ξ_4 , τ_0 and r, ω which can be solved by using the Maple software package to get the following results:

$$\begin{aligned} r = & -\frac{\omega^3}{9p^2} \{ 19\alpha_3^4 \alpha_4 + 46\alpha_3^3 \alpha_2^2 \alpha_4 + 46\alpha_3^2 \alpha_2^3 \alpha_4 - 24\alpha_3 \alpha_2 \alpha_4^4 + 56\alpha_3 \alpha_4^3 \alpha_2^2 - 52\alpha_3 \alpha_4^2 \alpha_2^3 \\ & + 19\alpha_3 \alpha_4 \alpha_2^4 + 8\alpha_3^3 \alpha_4^3 - 4\alpha_3^2 \alpha_4^4 - \alpha_4 \alpha_3^5 - \alpha_3 \alpha_2^5 - \alpha_2 \alpha_3^5 - 8\alpha_3^4 \alpha_2^2 - 14\alpha_2^3 \alpha_3^3 - 8\alpha_2^2 \alpha_4^4 \\ & - \alpha_4 \alpha_2^5 - 4\alpha_4^4 \alpha_2^2 + 8\alpha_4^3 \alpha_2^3 - 3\alpha_4^2 \alpha_2^4 - 3\alpha_3^4 \alpha_2^4 - 52\alpha_3^3 \alpha_2^2 \alpha_4 + 56\alpha_3^2 \alpha_4^3 \alpha_2 - 82\alpha_3^2 \alpha_4^2 \alpha_2^2 \} \\ & - \frac{\omega^2 D}{9p^2} (\alpha_3^4 + 16\alpha_4^4 + \alpha_2^4 + 20\alpha_3^2 \alpha_4^2 - 32\alpha_3 \alpha_4^3 + 14\alpha_3^2 \alpha_2^2 - 32\alpha_2 \alpha_4^3 + 20\alpha_4^2 \alpha_2^2 - 4\alpha_4 \alpha_2^3 \\ & - 4\alpha_4 \alpha_3^3 + 56\alpha_3 \alpha_2 \alpha_4^2 - 28\alpha_3 \alpha_4 \alpha_2^2 - 28\alpha_3^2 \alpha_2 \alpha_4) + \frac{\omega q}{9p} (4\alpha_3 \alpha_4 + 6\alpha_2 \alpha_3 - 4\alpha_4^2 + \alpha_2^2 + 4\alpha_4 \alpha_2 + \\ & + \alpha_3^2) - \frac{2q}{3p} D, \quad \xi_3 = -2\zeta_0(\alpha_2 + \alpha_3), \quad \xi_4 = \zeta_0, \quad \alpha_1 = \alpha_2 + \alpha_3 - \alpha_4, \quad \tau_0 = \frac{D}{p}, \end{aligned} \tag{3.61}$$

where

$$\begin{aligned} D = & \frac{\omega}{6} (4\alpha_2 \alpha_4 + 4\alpha_4 \alpha_3 - 4\alpha_4^2 + 6\alpha_2 \alpha_3 + \alpha_2^2 + \alpha_3^2) \pm \frac{\omega}{6} [16\alpha_4^4 - 32\alpha_4^3 (\alpha_2 + \alpha_3) \\ & + \alpha_4^2 (20\alpha_2^2 + 20\alpha_3^2 + 56\alpha_2 \alpha_3) + \alpha_4 (-28\alpha_2 \alpha_3^2 - 28\alpha_2^2 \alpha_3 - 4\alpha_3^3 - 4\alpha_2^3) - \frac{12pq}{\omega^2} \\ & + \alpha_2^4 + \alpha_3^4 + 14\alpha_3^2 \alpha_2^2]^{1/2} \end{aligned}$$

Equations (3.61) , (3.34) and (3.35) lead to get:

$$\begin{aligned} \xi_0 = & \zeta_0 \alpha_2 \alpha_3 \alpha_4 (\alpha_2 + \alpha_3 - \alpha_4), \quad \xi_1 = -(\alpha_3 \alpha_4 - \alpha_4^2 + \alpha_2 \alpha_4 + \alpha_2 \alpha_3) (\alpha_2 + \alpha_3) \zeta_0, \\ \xi_2 = & \zeta_0 (3\alpha_2 \alpha_3 + \alpha_2 \alpha_4 + \alpha_3^2 + \alpha_3 \alpha_4 - \alpha_4^2 + \alpha_2^2), \quad \tau_2 = \frac{2\omega}{p}, \quad \tau_1 = \frac{-2\omega(\alpha_3 + \alpha_2)}{p}, \end{aligned} \tag{3.62}$$

where ζ_0 is an arbitrary constant and

$$\begin{aligned} \pm(\xi - \eta_0) &= \int \frac{dY}{\sqrt{(Y - (\alpha_2 + \alpha_3 - \alpha_4))(Y - \alpha_2)(Y - \alpha_3)(Y - \alpha_4)}} \\ &= -\frac{2}{(\alpha_2 - \alpha_4)} \text{EllipticF} \left[\sqrt{\frac{(\alpha_2 - \alpha_4)(Y - \alpha_4)}{(\alpha_2 + \alpha_3 - 2\alpha_4)(Y - \alpha_3)}}, \sqrt{\frac{(\alpha_2 - \alpha_3)(\alpha_2 + \alpha_3 - 2\alpha_4)}{(\alpha_4 - \alpha_2)^2}} \right] \end{aligned} \quad (3.62)$$

or

$$Y = \frac{\alpha_4^2 - \alpha_2\alpha_4 + (\alpha_2\alpha_3 + \alpha_3^2 - 2\alpha_3\alpha_4)sn^2\left(\frac{1}{2}(\alpha_2 - \alpha_4)(\xi - \eta_0), \sqrt{\frac{(\alpha_2 - \alpha_3)(\alpha_2 + \alpha_3 - 2\alpha_4)}{(\alpha_4 - \alpha_2)^2}}\right)}{\alpha_4 - \alpha_2 + (\alpha_2 + \alpha_3 - 2\alpha_4)sn^2\left(\frac{1}{2}(\alpha_2 - \alpha_4)(\xi - \eta_0), \sqrt{\frac{(\alpha_2 - \alpha_3)(\alpha_2 + \alpha_3 - 2\alpha_4)}{(\alpha_4 - \alpha_2)^2}}\right)} \quad (3.63)$$

Substituting the solutions (3.63), (3.62) and (3.61) into (3.43), we get the exact solution of Eqs.(3.4) takes the form:

$$\begin{aligned} v(\xi) &= \frac{D}{p} \\ &= -\frac{2\omega(\alpha_3 + \alpha_2)}{p} \left[\frac{\alpha_4^2 - \alpha_2\alpha_4 + (\alpha_2\alpha_3 + \alpha_3^2 - 2\alpha_3\alpha_4)sn^2\left(\frac{1}{2}(\alpha_2 - \alpha_4)(\xi - \eta_0), \sqrt{\frac{(\alpha_2 - \alpha_3)(\alpha_2 + \alpha_3 - 2\alpha_4)}{(\alpha_4 - \alpha_2)^2}}\right)}{\alpha_4 - \alpha_2 + (\alpha_2 + \alpha_3 - 2\alpha_4)sn^2\left(\frac{1}{2}(\alpha_2 - \alpha_4)(\xi - \eta_0), \sqrt{\frac{(\alpha_2 - \alpha_3)(\alpha_2 + \alpha_3 - 2\alpha_4)}{(\alpha_4 - \alpha_2)^2}}\right)} \right] \\ &+ \frac{2\omega}{p} \left[\frac{\alpha_4^2 - \alpha_2\alpha_4 + (\alpha_2\alpha_3 + \alpha_3^2 - 2\alpha_3\alpha_4)sn^2\left(\frac{1}{2}(\alpha_2 - \alpha_4)(\xi - \eta_0), \sqrt{\frac{(\alpha_2 - \alpha_3)(\alpha_2 + \alpha_3 - 2\alpha_4)}{(\alpha_4 - \alpha_2)^2}}\right)}{\alpha_4 - \alpha_2 + (\alpha_2 + \alpha_3 - 2\alpha_4)sn^2\left(\frac{1}{2}(\alpha_2 - \alpha_4)(\xi - \eta_0), \sqrt{\frac{(\alpha_2 - \alpha_3)(\alpha_2 + \alpha_3 - 2\alpha_4)}{(\alpha_4 - \alpha_2)^2}}\right)} \right]^2 \end{aligned} \quad (3.64)$$

Hence the exact solution of nonlinear Zhiber Shabat differential equation (3.1) take the following form:

$$\begin{aligned} u(\xi) &= \ln \left(\frac{D}{p} \right. \\ &= -\frac{2\omega(\alpha_3 + \alpha_2)}{p} \left[\frac{\alpha_4^2 - \alpha_2\alpha_4 + (\alpha_2\alpha_3 + \alpha_3^2 - 2\alpha_3\alpha_4)sn^2\left(\frac{1}{2}(\alpha_2 - \alpha_4)(\xi - \eta_0), \sqrt{\frac{(\alpha_2 - \alpha_3)(\alpha_2 + \alpha_3 - 2\alpha_4)}{(\alpha_4 - \alpha_2)^2}}\right)}{\alpha_4 - \alpha_2 + (\alpha_2 + \alpha_3 - 2\alpha_4)sn^2\left(\frac{1}{2}(\alpha_2 - \alpha_4)(\xi - \eta_0), \sqrt{\frac{(\alpha_2 - \alpha_3)(\alpha_2 + \alpha_3 - 2\alpha_4)}{(\alpha_4 - \alpha_2)^2}}\right)} \right] \\ &+ \frac{2\omega}{p} \left[\frac{\alpha_4^2 - \alpha_2\alpha_4 + (\alpha_2\alpha_3 + \alpha_3^2 - 2\alpha_3\alpha_4)sn^2\left(\frac{1}{2}(\alpha_2 - \alpha_4)(\xi - \eta_0), \sqrt{\frac{(\alpha_2 - \alpha_3)(\alpha_2 + \alpha_3 - 2\alpha_4)}{(\alpha_4 - \alpha_2)^2}}\right)}{\alpha_4 - \alpha_2 + (\alpha_2 + \alpha_3 - 2\alpha_4)sn^2\left(\frac{1}{2}(\alpha_2 - \alpha_4)(\xi - \eta_0), \sqrt{\frac{(\alpha_2 - \alpha_3)(\alpha_2 + \alpha_3 - 2\alpha_4)}{(\alpha_4 - \alpha_2)^2}}\right)} \right]^2 \left. \right) \end{aligned} \quad (3.65)$$

Family 8. If equation (3.46) has four complex roots $\alpha_1 = N_1 + iN_2$, $\alpha_2 = N_1 - iN_2$, $\alpha_3 = N_3 + iN_4$, $\alpha_4 = N_3 - iN_4$, $N_j, j=1, \dots, 4$ are real numbers, consequently we can write the (3.46) in the following form:

$$\begin{aligned} & \frac{\xi_4}{\zeta_0} Y^4 + \frac{\xi_3}{\zeta_0} Y^3 + \frac{1}{4\xi_4 \omega^2 \zeta_0 (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[-12\omega^2 \xi_3^4 \tau_0 p \xi_4 \zeta_0 + \omega^3 \xi_3^6 - 16\xi_4^2 p \zeta_0^2 \omega \xi_3^2 q \right. \\ & + 128\xi_4^3 p^2 \zeta_0^3 q \tau_0 + 256\xi_4^3 p^3 \zeta_0^3 \tau_0^3 + 64\xi_4^3 p^2 \zeta_0^3 r \left. \right] Y^2 + \frac{\xi_3 p}{2\xi_4 \omega (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[\omega^2 \tau_0 \xi_3^4 \right. \\ & - 4\omega \xi_3^2 q \xi_4 \zeta_0 - 16\omega \xi_3^2 \xi_4 \tau_0^2 p \zeta_0 + 32q \tau_0 \zeta_0^2 p \xi_4^2 + 64p^2 \tau_0^3 \zeta_0^2 \xi_4^2 + 16r \zeta_0^2 \xi_4^2 p \left. \right] Y \\ & + \frac{p^2 \zeta_0^2}{4\xi_4 \omega^2 (-\omega \xi_3^2 + 8\tau_0 \xi_4 p \zeta_0)^2} \left[\omega^2 \tau_0^2 \xi_3^4 - 16\omega \tau_0^3 \xi_3^2 \xi_4 p \zeta_0 - 8\tau_0 \xi_4 \zeta_0 \omega \xi_3^2 q + 64\tau_0^2 \xi_4^2 p \zeta_0^2 q \right. \\ & + 64\tau_0^4 \xi_4^2 p^2 \zeta_0^2 + 64\tau_0 \xi_4^2 p \zeta_0^2 r - 4\omega \xi_3^2 r \zeta_0 \xi_4 \left. \right] \\ & - (Y - (N_1 + iN_2))(Y - (N_1 - iN_2))(Y - (N_3 + iN_4))(Y - (N_3 - iN_4)) = 0. \end{aligned} \tag{3.66}$$

From equating the coefficients of Y of both sides of Eq.(3.66), we get a system of algebraic equations in $\zeta_0, \xi_3, \xi_4, \tau_0$ and r, ω which can be solved by using the Maple software package to get the following results:

$$\begin{aligned} r = & \frac{2D}{9p^2} (-8\omega^2 N_2^4 + 8\omega^2 N_2^2 N_4^2 - 8\omega^2 N_4^4 - 3pq) + \frac{2}{9p^2} \{ 16\omega^3 N_2^4 N_3^2 - 16\omega^3 N_2^2 N_4^2 N_3^2 \\ & + 8\omega^3 N_2^2 N_4^4 + 8\omega^3 N_2^4 N_4^2 + 6pq\omega N_3^2 + 16\omega^3 N_4^4 N_3^2 + 2\omega pq N_2^2 + 2\omega pq N_4^2 \} \\ & + \alpha_3^2 - \frac{2q}{3p} D, \quad \xi_3 = -4\zeta_0 N_3, \quad \xi_4 = \zeta_0, \quad N_1 = N_3, \quad \tau_0 = \frac{D}{p}, \end{aligned} \tag{3.67}$$

where $D = \frac{\omega}{3} (6N_3^2 + 2N_4^2 + 2N_2^2) \pm \frac{1}{3} \sqrt{4\omega^2 N_4^4 - 4\omega^2 N_2^2 N_4^2 + 4\omega^2 N_2^4 - 3pq}$.

Equations (3.94), (3.64) and (3.65) lead to get:

$$\begin{aligned} \xi_0 = & (N_2^2 N_3^2 + N_2^2 N_4^2 + N_3^4 + N_4^2 N_3^2) \zeta_0, \quad \xi_1 = -2(2N_3^2 + N_4^2 + N_2^2) N_3 \zeta_0, \\ \xi_2 = & (6N_3^2 + N_4^2 + N_2^2) \zeta_0, \quad \xi_3 = -4N_3 \zeta_0, \\ \tau_2 = & \frac{2\omega}{p}, \quad \tau_1 = \frac{-4\omega N_3}{p} \end{aligned} \tag{3.68}$$

where ζ_0 is an arbitrary constant and

$$\begin{aligned} \pm (\xi - \eta_0) = & \int \frac{dY}{\sqrt{(Y^2 - 2N_3 Y + N_3^2 + N_2^2)(Y^2 - 2N_3 Y + N_3^2 + N_4^2)}} \\ = & \frac{2}{(N_2 - N_4)} \text{EllipticF} \left[\sqrt{\frac{(N_2 - N_4)(-Y + N_3 + iN_4)}{(N_2 + N_4)(-Y + N_3 - iN_4)}}, \frac{(N_2 + N_4)}{(N_2 - N_4)} \right] \end{aligned}$$

$$(3.69)$$

or

$$Y = \frac{(N_4 - N_2)(N_3 + iN_4) + (N_4 + N_2)(N_3 - iN_4)sn^2\left(\frac{1}{2}(N_2 - N_4)(\xi - \eta_0), \frac{(N_2 + N_4)}{(N_2 - N_4)}\right)}{(N_4 - N_2) + (N_4 + N_2)sn^2\left(\frac{1}{2}(N_2 - N_4)(\xi - \eta_0), \frac{(N_2 + N_4)}{(N_2 - N_4)}\right)} \quad (3.70)$$

Substituting the solutions (3.70), (3.68) and (3.67) into (3.43), we get the exact solution of Eqs.(3.4) takes the form:

$$v(\xi) = -\frac{4\omega N_3}{p} \left[\frac{(N_4 - N_2)(N_3 + iN_4) + (N_4 + N_2)(N_3 - iN_4)sn^2\left(\frac{1}{2}(N_2 - N_4)(\xi - \eta_0), \frac{(N_2 + N_4)}{(N_2 - N_4)}\right)}{(N_4 - N_2) + (N_4 + N_2)sn^2\left(\frac{1}{2}(N_2 - N_4)(\xi - \eta_0), \frac{(N_2 + N_4)}{(N_2 - N_4)}\right)} \right] + \frac{2\omega}{p} \left[\frac{(N_4 - N_2)(N_3 + iN_4) + (N_4 + N_2)(N_3 - iN_4)sn^2\left(\frac{1}{2}(N_2 - N_4)(\xi - \eta_0), \frac{(N_2 + N_4)}{(N_2 - N_4)}\right)}{(N_4 - N_2) + (N_4 + N_2)sn^2\left(\frac{1}{2}(N_2 - N_4)(\xi - \eta_0), \frac{(N_2 + N_4)}{(N_2 - N_4)}\right)} \right]^2 + \frac{D}{p} \quad (3.71)$$

Hence the exact solution of nonlinear Zhiber Shabat differential equation (3.1) take the following form:

$$u(x, t) = \ln \left(-\frac{4\omega N_3}{p} \left[\frac{(N_4 - N_2)(N_3 + iN_4) + (N_4 + N_2)(N_3 - iN_4)sn^2\left(\frac{1}{2}(N_2 - N_4)(x - \omega t - \eta_0), \frac{(N_2 + N_4)}{(N_2 - N_4)}\right)}{(N_4 - N_2) + (N_4 + N_2)sn^2\left(\frac{1}{2}(N_2 - N_4)(x - \omega t - \eta_0), \frac{(N_2 + N_4)}{(N_2 - N_4)}\right)} \right] + \frac{2\omega}{p} \left[\frac{(N_4 - N_2)(N_3 + iN_4) + (N_4 + N_2)(N_3 - iN_4)sn^2\left(\frac{1}{2}(N_2 - N_4)(x - \omega t - \eta_0), \frac{(N_2 + N_4)}{(N_2 - N_4)}\right)}{(N_4 - N_2) + (N_4 + N_2)sn^2\left(\frac{1}{2}(N_2 - N_4)(x - \omega t - \eta_0), \frac{(N_2 + N_4)}{(N_2 - N_4)}\right)} \right]^2 + \frac{D}{p} \right) \quad (3.72)$$

4. Conclusion

In this paper, we used the extended trial equation method to construct a series of some new analytic exact solutions for some nonlinear partial differential equations in mathematical physics when the balance numbers is positive integer. We constructed the exact solutions in many different functions such as hyperbolic function solutions, trigonometric function solutions, Jacobi elliptic functions solutions and rational solutions for the nonlinear Zhiber Shabat nonlinear differential equations.. This method is more powerful than other method for solving the nonlinear partial differential equations. This method can be used to solve many nonlinear partial differential equations in mathematical physics.

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